ADVANCED ENGINEERING

Advanced Engineering Mathematics

A new edition of Further Engineering Mathematics

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with additions by

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Preface to the First Edition

The purpose of this book is essentially to provide a sound second year course in Mathematics appropriate to studies leading to B.Sc. Engineering Degrees and other qualifications of a comparable level. The emphasis throughout is on techniques and applications, supported by sufficient formal proofs to warrant the methods being employed.

The structure of the text and the techniques used follow closely those of the author's first year book, *Engineering Mathematics – Programmes and Problems*, to which this further book is a companion volume and a continuation of the highly successful learning strategies devised. As with the previous work, the text is based on a series of self-instructional programmes arising from extensive research and rigid evaluation in a variety of relevant courses and, once again, the individualised nature of the development makes the book eminently suitable both for general class use and for personal study.

Each of the course programmes guides the student through the development of a particular topic, with numerous worked examples to demonstrate the techniques and with increased responsibility passing to the student as mastery is achieved. Revision exercises are provided where appropriate and each programme terminates with a *Revision Summary* of the main points covered, a *Test Exercise* based directly on the work of the programme and a set of *Further Problems* which provides opportunity for the additional practice that is essential for ensured success. The ability to work at one's own pace throughout is of utmost importance in maintaining motivation and in achieving mastery.

In several instances, the topic of a programme is a direct extension of basic work covered in *Engineering Mathematics* and where this is so, the title page of the programme carries a brief reference to the relevant programme in the first year treatment. This clearly directs the student to worthwhile revision of the prerequisites assumed in the further development of the subject matter.

A complete set of Answers to all problems and a detailed Index are provided at the end of the book.

Grateful acknowledgement is made of the constructive suggestions and cooperation received from many quarters both in the development of the original programmes and in the final preparation of the text. Recognition must also be made of the many sources from which

examples have been gleaned over the years and which contribute in no small measure to the success of the work.

Finally my sincere appreciation is due to the publishers for their patience, advice and ready cooperation in the preparation of the text for publication.

K.A. Stroud

Preface to the Second Edition

Since the first publication of *Further Engineering Mathematics* as core material for a typical second year engineering degree course, requests have been received from time to time for the inclusion of further topics to cover the particular requirements of individual syllabuses.

Some limit, inevitably, has to be placed on the physical size of the text, but it has been possible at least to include a programme on *Linear Optimisation* (*Linear Programming*) which was one of the subjects most frequently required.

The treatment of the additional material follows the structure of the rest of the book and the emphasis is largely on the practical use of the *simplex method* for the solution of both maximisation and minimisation problems.

The opportunity has also been taken to amend and clarify a number of minor points in the existing text and my thanks are due to those correspondents who have undertaken to write with constructive comment. Such feedback is always welcome.

K.A.S.

Preface to the Third Edition

With the new edition of *Further Engineering Mathematics*, the opportunity has been taken to incorporate a number of minor revisions and amendments to the previous text.

The format of the pages has been changed and the publishers have undertaken the complete resetting of the text to result in a more open presentation of the material and to facilitate the learning process still further.

Once again, my sincere thanks are due to all those correspondents who have kindly written with constructive comment concerning the book and to the publishers for their continued support, advice and cooperation throughout the preparation, production and marketing of the work.

K.A.S.

Preface to the Fourth Edition

It is now nearly 20 years since *Advanced Engineering Mathematics* (in earlier editions called *Further Engineering Mathematics*) by Ken Stroud was published and from the start it has been one of the most widely used and successful textbooks for science and engineering students at this level. I am delighted to have been asked to contribute to a new edition. As with the fifth edition of *Engineering Mathematics* I have endeavoured to retain the very essence of the book that has contributed to so many students' mathematical abilities over the years, particularly the time-tested Stroud format with its close attention to technique development throughout. In my task I have been greatly assisted by a first-rate team of academics who have worked alongside me in the development of this edition. To them I should like to express my sincere gratitude for all the detailed care and consideration they have given to all my contributions.

Immediately noticeable is the title change from *Further Engineering Mathematics* to *Advanced Engineering Mathematics* which, it is felt, more clearly describes the contents to a world-wide audience. Because a substantial amount of material in the first two Programmes of the earlier editions is no longer taught in the detail given, the first significant change to the contents has been their consolidation into a single Programme called *Numerical solutions of equations and interpolation*. To cater for continual changes in engineering mathematics four new Programmes have been added: *Z transforms, Introduction to the Fourier transform, Numerical solutions of partial differential equations* and *Complex analysis 3*, the last dealing with complex integration. The two original Programmes dealing with the *Laplace transform* have been separated into three Programmes with the addition of new material on harmonic oscillators. Sturm–Liouville systems have been introduced into the Programme *Power series solutions of ordinary differential equations* and predictor–corrector methods have been added to the Programme *Numerical solutions of ordinary differential equations*.

To follow the format of the fifth edition of *Engineering Mathematics* and to give as much assistance as possible in organising the student's study I have introduced specific **Learning outcomes** at the beginning and **Can You?** checklists at the end of each Programme. In this way the learning experience is made more explicit and the student is given greater confidence in what has been learnt.

It is only in working on this new edition, just as with the earlier book *Engineering Mathematics*, that the enormity of Ken Stroud's achievement can be really understood. The vast amount of work involved, the care and attention to detail and above all the complete understanding of his students and their learning processes are apparent in every page. It has been both a challenge and an honour to be able to work on such a book. I should like to thank the Stroud family again for their support in my work for this new edition. I should also like to thank my Editor, Helen Bugler, and her erstwhile assistant, Esther Thackeray, for their continued good humour, care and professionalism that have been invaluable in the creation of this new edition.

Huddersfield
February 2003

Dexter J. Booth

Hints on using the Book

This book contains twenty-three Programmes, each of which has been written in such a way as to make learning more effective and more interesting. It is almost like having a personal tutor, for you proceed at your own rate of learning and any difficulties you may have are cleared before you have the chance to practise incorrect ideas or techniques.

You will find that each Programme is divided into sections called frames. When you start a Programme, begin at Frame 1. Read each frame carefully and carry out any instructions or exercise which you are asked to do. In almost every frame, you are required to make a response of some kind, testing your understanding of the information in the frame, and you can immediately compare your answer with the correct answer given in the next frame. To obtain the greatest benefit, you are strongly advised to cover up the following frame, where necessary, until you have made your response. When a series of dots occurs, you are expected to supply the missing word, phrase, or number. At every stage, you will be guided along the right path. There is no need to hurry: read the frames carefully and follow the directions exactly. In this way, you must learn.

At the end of each Programme, you will find a **Revision summary** and a **Can You?** checklist that matches the **Learning outcomes** given at the beginning of the Programme. Read these carefully to make sure you have not missed anything. Next you will find a short **Test exercise**. This is set directly on what you have learned in the Programme: the questions are straightforward and contain no tricks. When you have completed these, return to the **Can You?** checklist as a final reminder of the contents of the Programme. To provide you with the necessary practice, a set of **Further problems** is also included. Remember that in mathematics, as in many other situations, practice makes perfect – or more nearly so.

Even if you feel you have done some of the topics before, work steadily through each Programme: it will serve as useful revision and fill in any gaps in your knowledge that you may have.

Useful background information

1 Algebraic identities

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a-b)^2 = a^2 - 2ab + b^2 \qquad (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$$

$$a^2 - b^2 = (a-b)(a+b)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

2 Trigonometrical identities

(1) $\sin^2\theta + \cos^2\theta = 1$; $\sec^2\theta = 1 + \tan^2\theta$;
$\operatorname{cosec}^2\theta = 1 + \cot^2\theta$

(2)
$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$
$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$
$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

(3) Let $A = B = \theta$ $\quad\therefore\quad \sin 2\theta = 2\sin\theta\cos\theta$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

(4) Let $\theta = \dfrac{\phi}{2}$ $\quad\therefore\quad \sin\phi = 2\sin\dfrac{\phi}{2}\cos\dfrac{\phi}{2}$

$$\cos\phi = \cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2} = 1 - 2\sin^2\frac{\phi}{2} = 2\cos^2\frac{\phi}{2} - 1$$

$$\tan\phi = \frac{2\tan\dfrac{\phi}{2}}{1-\tan^2\dfrac{\phi}{2}}$$

(5) $\sin C + \sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$

$\sin C - \sin D = 2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

$\cos C + \cos D = 2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$

$\cos D - \cos C = 2\sin\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

(6) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

(7) Negative angles: $\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

(8) Angles having the same trigonometrical ratios:

(a) Same sine: θ and $(180° - \theta)$

(b) Same cosine: θ and $(360° - \theta)$, i.e. $(-\theta)$

(c) Same tangent: θ and $(180° + \theta)$

(9) $a\sin\theta + b\cos\theta = A\sin(\theta+\alpha)$

$a\sin\theta - b\cos\theta = A\sin(\theta-\alpha)$

$a\cos\theta + b\sin\theta = A\cos(\theta-\alpha)$

$a\cos\theta - b\sin\theta = A\cos(\theta+\alpha)$

where $\begin{cases} A = \sqrt{a^2+b^2} \\ \alpha = \tan^{-1}\dfrac{b}{a} \quad (0° < \alpha < 90°) \end{cases}$

3 Standard curves

(a) *Straight line*

Slope, $m = \dfrac{dy}{dx} = \dfrac{y_2 - y_1}{x_2 - x_1}$

Angle between two lines, $\tan\theta = \dfrac{m_2 - m_1}{1 + m_1 m_2}$

For parallel lines, $m_2 = m_1$

For perpendicular lines, $m_1 m_2 = -1$

Equation of a straight line (slope $= m$)

(1) Intercept c on real y-axis: $y = mx + c$

(2) Passing through (x_1, y_1): $y - y_1 = m(x - x_1)$

(3) Joining (x_1, y_1) and (x_2, y_2): $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$

(b) *Circle*

Centre at origin, radius r: $x^2 + y^2 = r^2$

Centre (h, k), radius r: $(x - h)^2 + (y - k)^2 = r^2$

General equation: $x^2 + y^2 + 2gx + 2fy + c = 0$

with centre $(-g, -f)$: radius $= \sqrt{g^2 + f^2 - c}$

Parametric equations: $x = r\cos\theta,\ y = r\sin\theta$

(c) *Parabola*

Vertex at origin, focus $(a, 0)$: $y^2 = 4ax$

Parametric equations: $x = at^2,\ y = 2at$

(d) *Ellipse*

Centre at origin, foci $\left(\pm\sqrt{a^2 + b^2}, 0\right)$: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

where $a =$ semi-major axis, $b =$ semi-minor axis

Parametric equations: $x = a\cos\theta,\ y = b\sin\theta$

(e) *Hyperbola*

Centre at origin, foci $\left(\pm\sqrt{a^2 + b^2}, 0\right)$: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

Parametric equations: $x = a\sec\theta,\ y = b\tan\theta$

Rectangular hyperbola:

Centre at origin, vertex $\pm\left(\dfrac{a}{\sqrt{2}}, \dfrac{a}{\sqrt{2}}\right)$: $xy = \dfrac{a^2}{2} = c^2$

where $c = \dfrac{a}{\sqrt{2}}$ i.e. $xy = c^2$

Parametric equations: $x = ct,\ y = c/t$

4 Laws of mathematics

(a) *Associative laws* – for addition and multiplication

$a + (b + c) = (a + b) + c$

$a(bc) = (ab)c$

(b) *Commutative laws* – for addition and multiplication

$a + b = b + a$

$ab = ba$

(c) *Distributive laws* – for multiplication and division

$a(b + c) = ab + ac$

$\dfrac{b + c}{a} = \dfrac{b}{a} + \dfrac{c}{a}$ (provided $a \neq 0$)

Numerical solutions of equations and interpolation

Frames 1 to 87

Learning outcomes

When you have completed this Programme you will be able to:

- Appreciate the Fundamental Theorem of Algebra
- Find the two roots of a quadratic equation and recognise that for polynomial equations with real coefficients complex roots exist in complex conjugate pairs
- Use the relationships between the coefficients and the roots of a polynomial equation to find the roots of the polynomial
- Transform a cubic equation to its reduced form
- Use Tartaglia's solution to find the real root of a cubic equation
- Find the solution of the equation $f(x) = 0$ by the method of bisection
- Solve equations involving a single real variable by iteration and use a spreadsheet for efficiency
- Solve equations using the Newton–Raphson iterative method
- Use the modified Newton–Raphson method to find the first approximation when the derivative is small
- Understand the meaning of *interpolation* and use simple linear and graphical interpolation
- Use the Gregory–Newton interpolation formula with forward and backward differences for equally spaced domain points
- Use the Gauss interpolation formulas using central differences for equally spaced domain points
- Use Lagrange interpolation when the domain points are not equally spaced

Introduction

1 In this Programme we shall be looking at analytic and numerical methods of solving the general equation in a single variable, $f(x) = 0$. In addition, a functional relationship can be exhibited in the form of a collection of ordered pairs rather than in the form of an algebraic expression. We shall be looking at interpolation methods of estimating values of $f(x)$ for intermediate values of x between those listed among the ordered pairs.

First we shall look at the **Fundamental Theorem of Algebra**, which deals with the factorisation of polynomials.

The Fundamental Theorem of Algebra

2 The *Fundamental Theorem of Algebra* can be stated as follows

Every polynomial expression $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ can be written as a product of n linear factors in the form

$$f(x) = a_n(x - r_1)(x - r_2)(\cdots)(x - r_n)$$

As an immediate consequence of this we can see that there are n values of x that satisfy the polynomial equation $f(x) = 0$, namely $x = r_1$, $x = r_2, \ldots, x = r_n$. We call these values the *roots* of the polynomial, but be aware that they may not all be distinct. Furthermore, the polynomial coefficients a_i and the polynomial roots r_i may be real, imaginary or complex.

For example the quadratic equation

$x^2 + 5x + 6 = 0$ can be written $(x + 2)(x + 3) = 0$ so it has the two *distinct* roots $x = -2$ and $x = -3$

$x^2 - 4x + 4 = 0$ can be written as $(x - 2)(x - 2) = 0$ so it has the two *coincident* roots $x = 2$ and $x = 2$

$x^2 + x + 1 = 0$ can be written as $(x + a)(x + b) = 0$ so it has the two roots $x = -a$ and $x = -b$

To find the numerical values of a and b we need to use the formula for finding the roots of a general quadratic equation. Can you recall what it is? If not, then refer to Frame 14 of Programme F.6 in *Engineering Mathematics, Fifth Edition.*

The solution to the quadratic equation $ax^2 + bx + c = 0$ is ....

The answer is in the next frame

3

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

So the roots of $x^2 + x + 1 = 0$ are ....

Next frame

4

$$\boxed{x = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}}$$

Because

$$a = b = c = 1 \text{ and so } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4}}{2}$$
$$= -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

This quadratic equation has two distinct *complex* roots. Notice that the two roots form a *complex conjugate pair* – each is the complex conjugate of the other. **Whenever a polynomial with *real coefficients* a_i has a complex root it also has the complex conjugate as another root.**

So given that $x = -2 + j\sqrt{5}$ is one root of a quadratic equation with real coefficients then

the other root is ....

5

$$\boxed{x = -2 - j\sqrt{5}}$$

Because

The complex conjugate of $x = -2 + j\sqrt{5}$ is $x = -2 - j\sqrt{5}$ and complex roots of a polynomial equation with real coefficients always appear as conjugate pairs.

The quadratic equation with these two roots is ....

6

$$\boxed{x^2 + 4x + 9 = 0}$$

Because

If $x = a$ and $x = b$ are the two roots of a quadratic equation then $(x - a)(x - b) = 0$ gives the quadratic equation. That is $(x - a)(x - b) = x^2 - (a + b)x + ab = 0$.

Here, the two roots are $x = -2 + j\sqrt{5}$ and $x = -2 - j\sqrt{5}$ so that

$$\left(x - \left[-2 + j\sqrt{5}\right]\right)\left(x - \left[-2 - j\sqrt{5}\right]\right) = 0$$

That is $x^2 - x\left[-2 + j\sqrt{5} - 2 - j\sqrt{5}\right] + \left[-2 + j\sqrt{5}\right]\left[-2 - j\sqrt{5}\right] = 0$.

So $x^2 + 4x + 9 = 0$.

Notice that the coefficients are ....

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$$\boxed{\text{Real}}$$

Relations between the coefficients and the roots of a polynomial equation

Let α, β, γ be the roots of $x^3 + px^2 + qx + r = 0$. Then, writing the expression $x^3 + px^2 + qx + r$ in terms of α, β, γ gives

$$x^3 + px^2 + qx + r = \ldots\ldots\ldots\ldots$$

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$$\boxed{(x - \alpha)(x - \beta)(x - \gamma)}$$

Therefore

$$\begin{aligned}
x^3 + px^2 + qx + r &= (x - \alpha)(x - \beta)(x - \gamma) \\
&= \left(x^2 - [\alpha + \beta]x + \alpha\beta\right)(x - \gamma) \\
&= x^3 - (\alpha + \beta)x^2 + \alpha\beta x - \gamma x^2 + (\alpha + \beta)\gamma x - \alpha\beta\gamma \\
&= x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma
\end{aligned}$$

$\therefore$ equating coefficients

(a) $\alpha + \beta + \gamma = \ldots\ldots\ldots\ldots$

(b) $\alpha\beta + \beta\gamma + \gamma\alpha = \ldots\ldots\ldots\ldots$

(c) $\alpha\beta\gamma = \ldots\ldots\ldots\ldots$

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$$\boxed{\text{(a)} \;\; -p; \quad \text{(b)} \;\; q; \quad \text{(c)} \;\; -r}$$

This, of course, applies to a cubic equation. Let us extend this to a more general equation.

So on to the next frame

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In general, if $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n$ are roots of the equation

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n = 0 \quad (p_0 \neq 0)$$

then sum of the roots $= -\dfrac{p_1}{p_0}$

sum of products of the roots, two at a time $= \dfrac{p_2}{p_0}$

sum of products of the roots, three at a time $= -\dfrac{p_3}{p_0}$

sum of products of the roots, n at a time $= (-1)^n . \dfrac{p_n}{p_0}$

So for the equation $3x^4 + 2x^3 + 5x^2 + 7x - 4 = 0$, if $\alpha, \beta, \gamma, \delta$ are the four roots, then

(a) $\alpha + \beta + \gamma + \delta =$ ....

(b) $\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \delta\beta + \gamma\alpha =$ ....

(c) $\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \alpha\beta\delta =$ ....

(d) $\alpha\beta\gamma\delta =$ ....

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(a) $-\dfrac{2}{3}$; (b) $\dfrac{5}{3}$; (c) $-\dfrac{7}{3}$; (d) $-\dfrac{4}{3}$

Now for a problem or two on the same topic.

Example 1

Solve the equation $x^3 - 8x^2 + 9x + 18 = 0$ given that the sum of two of the roots is 5.

Using the same approach as before, if α, β, γ are the roots, then

(a) $\alpha + \beta + \gamma =$ ....

(b) $\alpha\beta + \beta\gamma + \gamma\alpha =$ ....

(c) $\alpha\beta\gamma =$ ....

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(a) 8; (b) 9; (c) -18

So we have $\alpha + \beta + \gamma = 8$ Let $\alpha + \beta = 5$

$\therefore\ 5 + \gamma = 8 \quad \therefore\ \gamma = 3$

Also $\alpha\beta\gamma = -18 \quad \alpha\beta(3) = -18 \quad \therefore\ \alpha\beta = -6$

$\alpha + \beta = 5 \quad \therefore\ \beta = 5 - \alpha \quad \therefore\ \alpha(5 - \alpha) = -6$

$\alpha^2 - 5\alpha - 6 = 0 \quad \therefore\ (\alpha - 6)(\alpha + 1) = 0 \quad \therefore\ \alpha = -1 \text{ or } 6$

$\therefore\ \beta = 6 \text{ or } -1$

Roots are $x = -1, 3, 6$

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Example 2

Solve the equation $2x^3 + 3x^2 - 11x - 6 = 0$ given that the three roots form an arithmetic sequence.

Let us represent the roots by $(a-k)$, a, $(a+k)$

Then the sum of the roots $= 3a = \ldots\ldots\ldots\ldots$

and the product of the roots $= a(a-k)(a+k) = \ldots\ldots\ldots\ldots$

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$$\boxed{3a = -\frac{3}{2}; \quad a(a+k)(a-k) = \frac{6}{2} = 3}$$

$$\therefore\ a = -\frac{1}{2} \qquad -\frac{1}{2}\left(\frac{1}{4} - k^2\right) = 3 \qquad \therefore\ k = \pm\frac{5}{2}$$

$$\text{If } k = \frac{5}{2} \quad a = -\frac{1}{2}; \quad a-k = -3; \quad a+k = 2$$

$$\text{If } k = -\frac{5}{2} \quad a = -\frac{1}{2}; \quad a-k = 2; \quad a+k = -3$$

$\therefore$ required roots are $-3, -\frac{1}{2}, 2$

Here is a similar one.

Example 3

Solve the equation $x^3 + 3x^2 - 6x - 8 = 0$ given that the three roots are in geometric sequence.

This time, let the roots be $\frac{a}{k}$, a, ak

Then $\frac{a}{k} = a + ak = \ldots\ldots\ldots\ldots$ and $\left(\frac{a}{k}\right)(a)(ak) = \ldots\ldots\ldots\ldots$

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$$\boxed{\text{sum of roots} = -3; \quad \text{product of roots} = 8}$$

It then follows that the roots are $\ldots\ldots\ldots\ldots, \ldots\ldots\ldots\ldots, \ldots\ldots\ldots\ldots$

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$$\boxed{-4, \quad 2, \quad -1}$$

The working rests on the relationships between the roots and the coefficients, i.e. if α, β, γ are the roots of the cubic equation

$$ax^3 + bx^2 + cx + d = 0$$

then (a) $\alpha + \beta + \gamma = \ldots\ldots\ldots\ldots$

(b) $\alpha\beta + \beta\gamma + \gamma\alpha = \ldots\ldots\ldots\ldots$

(c) $\alpha\beta\gamma = \ldots\ldots\ldots\ldots$

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$$\boxed{\text{(a)}\ -\frac{b}{a};\quad \text{(b)}\ \frac{c}{a};\quad \text{(c)}\ -\frac{d}{a}}$$

In each of the three examples reconstruct the cubic to confirm that they are correct.

Now on to the next stage

Cubic equations

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The Fundamental Theorem of Algebra tells us that every cubic expression

$$f(x) = ax^3 + bx^2 + cx + d$$

can be written as a product of three linear factors

$$f(x) = a(x - r_1)(x - r_2)(x - r_3)$$

Consequently, every cubic equation

$$f(x) = a(x - r_1)(x - r_2)(x - r_3) = 0$$

has three roots which may be distinct or coincident and which may be real or complex. However, because complex roots of a polynomial with real coefficients always appear in complex conjugate pairs we can say that every such cubic equation has

at least one ....

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$$\boxed{\text{at least one real root}}$$

To find the value of this real root we can employ a formula equivalent to the formula used to find the two roots of the general quadratic. This is called Tartaglia's method but before we can proceed to look at that we must first consider how to transform the general cubic to its **reduced form**.

Next frame

Transforming a cubic to reduced form

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In every case, an equation of the form

$$x^3 + ax^2 + bx + c = 0$$

can be converted into the reduced form $y^3 + py + q = 0$ by the substitution $x = y - \frac{a}{3}$.

The example overleaf will demonstrate the method. ▶

Example 4

Express $f(x) = x^3 + 6x^2 - 4x + 5 = 0$ in reduced form.

Substitute $x = y - \frac{a}{3}$, i.e. $x = y - \frac{6}{3} = y - 2$. Put $x = y - 2$.

The equation then becomes

$$(y-2)^3 + 6(y-2)^2 - 4(y-2) + 5 = 0$$
$$(y^3 - 3y^2 2 + 3y4 - 8) + 6(y^2 - 4y + 4) - 4(y-2) + 5 = 0$$

which simplifies to ....

21

$$\boxed{y^3 - 16y + 29 = 0}$$

Tartaglia's solution for a real root

In the sixteenth century, Tartaglia discovered that a root of the cubic equation $x^3 + ax + b = 0$, where $a > 0$, is given by

$$x = \left\{-\frac{b}{2} + \sqrt{\frac{a^3}{27} + \frac{b^2}{4}}\right\}^{1/3} + \left\{-\frac{b}{2} - \sqrt{\frac{a^3}{27} + \frac{b^2}{4}}\right\}^{1/3}$$

That looks pretty formidable, but it is a good deal easier than it appears. Notice that $\frac{b}{2}$ and $\sqrt{\frac{a^3}{27} + \frac{b^2}{4}}$ occur twice and it is convenient to evaluate these first and then substitute the results in the main expression for x.

Example 5

Find a real root of $x^3 + 2x + 5 = 0$.

Here, $a = 2$, $b = 5$ $\therefore$ $\frac{b}{2} = 2{\cdot}5$

$$\sqrt{\frac{a^3}{27} + \frac{b^2}{4}} = \sqrt{\frac{8}{27} + \frac{25}{4}} = \sqrt{6{\cdot}5463} = 2{\cdot}5586$$

Then $x = (-2{\cdot}5 + 2{\cdot}5586)^{1/3} + (-2{\cdot}5 - 2{\cdot}5586)^{1/3}$

$$= 0{\cdot}3884 - 1{\cdot}7166 = -1{\cdot}3282 \qquad x = -1{\cdot}328$$

Once we have a real root, the equation can be reduced to a quadratic and the remaining two roots determined. They are $x = 0{\cdot}664 + j1{\cdot}823$ and $x = 0{\cdot}664 - j1{\cdot}823$ (see *Engineering Mathematics, Fifth Edition*, Programme F.6).

Example 6

Determine a real root of $2x^3 + 3x - 4 = 0$.

This is first written $x^3 + 1{\cdot}5x - 2 = 0$ $\therefore$ $a = 1{\cdot}5$, $b = -2$

Now you can evaluate $\frac{b}{2}$ and $\sqrt{\frac{a^3}{27} + \frac{b^2}{4}}$ and so determine

$$x = \ldots\ldots\ldots\ldots$$

22

$$\boxed{0{\cdot}8796}$$

Because

$$\left\{-\frac{b}{2}+\sqrt{\frac{a^3}{27}+\frac{b^2}{4}}\right\}^{1/3}=\{2{\cdot}06066\}^{1/3}=1{\cdot}2725 \text{ and}$$

$$\left\{-\frac{b}{2}-\sqrt{\frac{a^3}{27}+\frac{b^2}{4}}\right\}^{1/3}=\{-0{\cdot}6066\}^{1/3}=-0{\cdot}3929,$$

therefore $x = 1{\cdot}2725 - 0{\cdot}3929 = 0{\cdot}8796$

Note: If you wish to find the real root of a cubic using Tartaglia's method and $a < 0$ then just multiply the entire equation by -1.

Next frame

Numerical methods

23

The methods that we have used so far to solve quadratic equations and to find the real root of a cubic equation are called *analytic methods*. These analytic methods used straightforward algebraic techniques to develop a formula for the answer. The numerical value of the answer can then be found by simple substitution of numbers for the variables in the formula. Unfortunately, general polynomial equations of order five or higher cannot by solved by analytic methods. Instead, we must resort to what are termed *numerical methods*. The simplest method of finding the solution to the equation $f(x) = 0$ is the *bisection* method.

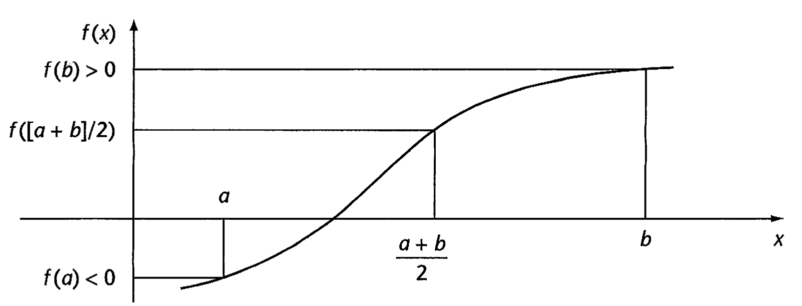
Bisection

The bisection method of finding a solution to the equation $f(x) = 0$ consists of

Finding a value of x, say $x = a$, such that $f(a) < 0$

Finding a value of x, say $x = b$, such that $f(b) > 0$

The solution to the equation $f(x) = 0$ must then lie between a and b. Furthermore, it must lie either in the first half of the interval between a and b or in the second half.



Find the value of $f([a+b]/2)$ – that is halfway between a and b.

If $f([a+b]/2) > 0$ then the solution lies in the first half and if $f([a+b]/2) < 0$ then it lies in the second half. This procedure is repeated, narrowing down the width of the interval by a half each time. An example should clarify all this.

Example 7

Find the positive value of x that satisfies the equation $x^2 - 2 = 0$.

Firstly we note that if $x = 1$ then $x^2 - 2 < 0$, and that if $x = 2$ then $x^2 - 2 > 0$, so the solution that we seek must lie between 1 and 2.

We look for the ....

24

The mid-point between 1 and 2 which is 1·5

Now, when $x = 1{\cdot}5$, $x^2 - 2 = 0{\cdot}25 > 0$

so the solution must lie between ....

25

1 and 1·5

The mid-point between 1 and 1·5 is 1·25. When $x = 1{\cdot}25$, $x^2 - 2 = -0{\cdot}4375 < 0$

so the solution must lie between ....

26

1·25 and 1·5

The mid-point between 1·25 and 1·5 is 1·375. We now evaluate $x^2 - 2$ at this point and determine in which half interval the solution lies. This process is repeated and the following table displays the results. In each block of six numbers the first column lists the end points of the interval and the mid-point. The second column contains the respective values $f(x) = x^2 - 2$. Construct the table as follows.

(a) For each block of six numbers copy the last number in the first column into the second place of the first column of the following block. This represents the centre point of the previous interval.

(b) For each block of six numbers copy the number that represents the other end point of the new interval from the first column into the first place of the first column of the following block. Look at the signs in the second column of the first block to decide which is the appropriate number.

▶

a	1·0000	−1·0000	1·0000	−1·0000	1·5000	0·2500	1·5000	0·2500
b	2·0000	2·0000	1·5000	0·2500	1·2500	−0·4375	1·3750	−0·1094
$(a+b)/2$	1·5000	0·2500	1·2500	−0·4375	1·3750	−0·1094	1·4375	0·0664
a	1·3750	−0·1094	1·4375	0·0664	1·4063	−0·0225	1·4219	0·0217
b	1·4375	0·0664	1·4063	−0·0225	1·4219	0·0217	1·4141	−0·0004
$(a+b)/2$	1·4063	−0·0225	1·4219	0·0217	1·4141	−0·0004	1·4180	0·0106
a	1·4141	−0·0004	1·4141	−0·0004	1·4141	−0·0004	1·4141	−0·0004
b	1·4180	0·0106	1·4160	0·0051	1·4150	0·0023	1·4146	0·0010
$(a+b)/2$	1·4160	0·0051	1·4150	0·0023	1·4146	0·0010	1·4143	0·0003
a	1·4141	−0·0004	1.4143	0·0003	1·4142	−0·0001		
b	1·4143	0·0003	1·4142	−0·0001	1·4142	0·0001		
$(a+b)/2$	1·4142	−0·0001	1·4142	0·0001	1·4142	0·0000		

The final result to four decimal places is $x = 1{\cdot}4142$ which is the correct answer to that level of accuracy – but it has taken a lot of activity to produce it. A much faster way of solving this equation is to use an iteration formula that was first devised by Newton.

Next frame

Numerical solution of equations by iteration

27

The process of finding the numerical solution to the equation

$$f(x) = 0$$

by iteration is performed by first finding an approximate solution and then using this approximate solution to find a more accurate solution. This process is repeated until a solution is found to the required level of accuracy. For example, Newton showed that the square root of a number a can be found from the iteration equation

$$x_{i+1} = \frac{1}{2}\left(x_i + \frac{a}{x_i}\right), \quad i = 0, 1, 2, \ldots$$

where x_0 is the approximation that starts the iteration off. So, to find a succession of approximate values of $\sqrt{2}$, each of increasing accuracy, we proceed as follows. Let $x_0 = 1{\cdot}5$ – found by the first stage of the bisection method. Then

$$x_1 = \frac{1}{2}\left(x_0 + \frac{a}{x_0}\right) = 0{\cdot}5(1{\cdot}5 + 2/1{\cdot}5) = 1{\cdot}4166\ldots$$

This value is then used to find x_2.

By rounding x_1 to 1·4167, the value of x_2 is found to be ....

28

$$\boxed{x_2 = 1{\cdot}4142}$$

Because

$$x_2 = \frac{1}{2}\left(x_1 + \frac{a}{x_1}\right) = 0{\cdot}5(1{\cdot}4167 + 2/1{\cdot}4167) = 1{\cdot}4142\ldots$$

This has achieved the same level of accuracy as the bisection method in just two steps.

Using a spreadsheet

This simple iteration procedure is more efficiently performed using a spreadsheet. If the use of a spreadsheet is a totally new experience for you then you are referred to Programme 4 of *Engineering Mathematics, Fifth Edition* where the spreadsheet is introduced as a tool for constructing graphs of functions. If you have a limited knowledge then you will be able to follow the text from here. The spreadsheet we shall be using here is Microsoft Excel, though all commercial spreadsheets possess the equivalent functionality.

Open your spreadsheet and in cell A1 enter n and press **Enter**. In this first column we are going to enter the iteration numbers. In cell A2 enter the number 0 and press **Enter**. Place the cell highlight in cell A2 and highlight the block of cells A2 to A7 by holding down the mouse button and wiping the highlight down to cell A7. Click the **Edit** command on the Command bar and point at **Fill** from the drop-down menu. Select **Series** from the next drop-down menu and accept the default **Step value** of 1 by clicking OK in the Series window.

The cells A3 to A7 fill with ....

29

the numbers 1 to 5

In cell B1 enter the letter x – this column is going to contain the successive x-values obtained by iteration. In cell B2 enter the value of x_0, namely 1·5.

In cell B3 enter the formula

= 0·5*(B2+2/B2)

The number that appears in cell B3 is then ....

30

1·416667

Place the cell highlight in cell B3, click the command **Edit** on the Command bar and select **Copy** from the drop-down menu. You have now copied the formula in cell B3 onto the Clipboard. Highlight the cells B4 to B7 and then click the **Edit** command again but this time select **Paste** from the drop-down menu.

The cells B4 to B7 fill with numbers to provide the display

....

31

n	x
0	1·5
1	1·416667
2	1·414216
3	1·414214
4	1·414214
5	1·414214

By using the various formatting facilities provided by the spreadsheet the display can be amended to provide the following

n	x
0	1·500000000000000
1	1·416666666666670
2	1·414215686274510
3	1·414213562374690
4	1·414213562373090
5	1·414213562373090

The number of decimal places here is 15, which is far greater than is normally required but it does demonstrate how effective a spreadsheet can be. In future we shall restrict the displays to 6 decimal places.

Notice that to find a value accurate to a given number of decimal places or significant figures it is sufficient to repeat the iterations until there is no change in the result from one iteration to the next.

Save your spreadsheet under some suitable name such as *Newton* because you may wish to use it again.

Now we shall look at this spreadsheet a little more closely

Relative addresses

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Place the cell highlight in cell B3 and the formula that it contains is =0·5*(B2+2/B2). Now place the cell highlight in cell B4 and the formula there is =0·5*(B3+2/B3). Why the difference?

When you enter the cell address B2 in the formula in B3 the spreadsheet understands that to mean *the contents of the cell immediately above*. It is this meaning that is copied into cell B4 where the *cell immediately above* is B3. If you wish to refer to a specific cell in a formula then you must use an **absolute address**.

Place the cell highlight in cell C1 and enter the number 2. Now place the cell highlight in cell B3 and re-enter the formula

= 0·5*(B2+C1/B2)

and copy this into cells B4 to B7. The numbers in the second column have not changed but the formulas have because in cells B3 to B7 the same reference is made to cell C1. *The use of the dollar signs has indicated an absolute address*. So why would we do this?

Change the number in cell C1 to 3 to obtain the display ....

33

n	x
0	1·500000000000000
1	1·750000000000000
2	1·732142857142860
3	1·732050810014730
4	1·732050807568880
5	1·732050807568880

These are the iterated values of $\sqrt{3}$ – the square root of the contents of cell C1. We can now use the same spreadsheet to find the square root of any positive number.

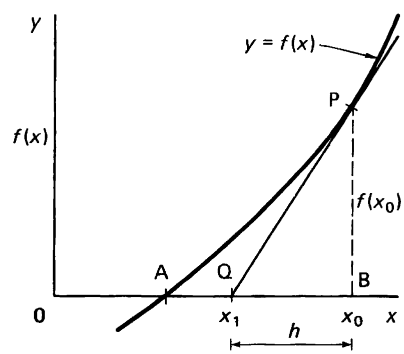
Newton's iterative procedure to find the square root of a positive number is a special case of the **Newton–Raphson** procedure to find the solution of the general equation $f(x) = 0$, and we shall look at this in the next frame.

Newton–Raphson iterative method

34

Consider the graph of $y = f(x)$ as shown. Then the x-value at the point A, where the graph crosses the x-axis, gives a solution of the equation $f(x) = 0$.

If P is a point on the curve near to A, then $x = x_0$ is an approximate value of the root of $f(x) = 0$, the error of the approximation being given by AB.



Let PQ be the tangent to the curve as P, crossing the x-axis at Q $(x_1, 0)$. Then $x = x_1$ is a better approximation to the required root.

From the diagram, $\dfrac{\text{PB}}{\text{QB}} = \left[\dfrac{dy}{dx}\right]_{\text{P}}$ i.e. the value of the derivative of y at the point P, $x = x_0$.

$$\therefore \quad \frac{\text{PB}}{\text{QB}} = f'(x_0) \quad \text{and} \quad \text{PB} = f(x_0)$$

$$\therefore \quad \text{QB} = \frac{\text{PB}}{f'(x_0)} = \frac{f(x_0)}{f'(x_0)} = h \text{ (say)}$$

$$x_1 = x_0 - h \qquad \therefore \; x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

▶

If we begin, therefore, with an approximate value (x_0) of the root, we can determine a better approximation (x_1). Naturally, the process can be repeated to improve the result still further. Let us see this in operation.

On to the next frame

35

Example 1

The equation $x^3 - 3x - 4 = 0$ is of the form $f(x) = 0$ where $f(1) < 0$ and $f(3) > 0$ so there is a solution to the equation between 1 and 3. We shall take this to be 2, by bisection. Find a better approximation to the root.

We have $f(x) = x^3 - 3x - 4 \quad \therefore \; f'(x) = 3x^2 - 3$

If the first approximation is $x_0 = 2$, then

$$f(x_0) = f(2) = -2 \quad \text{and} \quad f'(x_0) = f'(2) = 9$$

A better approximation x_1 is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0{}^3 - 3x_0 - 4}{3x_0{}^2 - 3}$$

$$x_1 = 2 - \frac{(-2)}{9} = 2{\cdot}22$$

$$\therefore \; x_0 = 2; \quad x_1 = 2{\cdot}22$$

If we now start from x_1 we can get a better approximation still by repeating the process.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{x_1{}^3 - 3x_1 - 4}{3x_1{}^2 - 3}$$

Here $x_1 = 2{\cdot}22 \qquad f(x_1) = \ldots\ldots\ldots\ldots; \quad f'(x_1) = \ldots\ldots\ldots\ldots$

36

$$\boxed{f(x_1) = 0{\cdot}281; \quad f'(x_1) = 11{\cdot}785}$$

Then $x_2 = \ldots\ldots\ldots\ldots$

37

$$\boxed{x_2 = 2{\cdot}196}$$

Because

$$x_2 = 2{\cdot}22 - \frac{0{\cdot}281}{11{\cdot}79} = 2{\cdot}196$$

Using $x_2 = 2{\cdot}196$ as a starter value, we can continue the process until successive results agree to the desired degree of accuracy.

$$x_3 = \ldots\ldots\ldots\ldots$$

38

$$\boxed{x_3 = 2{\cdot}196}$$

Because

$$f(x_2) = f(2{\cdot}196) = 0{\cdot}002026; \qquad f'(x_2) = f'(2{\cdot}196) = 11{\cdot}467$$

$$\therefore \; x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2{\cdot}196 - \frac{0{\cdot}00203}{11{\cdot}467} = 2{\cdot}196 \text{ (to 4 sig. fig.)}$$

The process is simple but effective and can be repeated again and again. Each repetition, or *iteration*, usually gives a result nearer to the required root $x = x_A$.

In general $x_{n+1} =$ ....

39

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Tabular display of results

Open your spreadsheet and in cells A1 to D1 enter the headings n, x, $f(x)$ and $f'(x)$

Fill cells A2 to A6 with the numbers 0 to 4

In cell B2 enter the value for x_0, namely 2

In cell C2 enter the formula for $f(x_0)$, namely =B2^3 – 3*B2 – 4 and copy into cells C3 to C6

In cell D2 enter the formula for $f'(x_0)$, namely =3*B2^2 – 3 and copy into cells D3 to D6

In cell B3 enter the formula for x_1, namely =B2 – C2/D2 and copy into cells B4 to B6.

The final display is ....

40

n	x	$f(x)$	$f'(x)$
0	2	−2	9
1	2·222222	0·30727	11·81481
2	2·196215	0·004492	11·47008
3	2·195823	1·01E-06	11·46492
4	2·195823	5·15E-14	11·46492

As soon as the number in the second column is repeated then we know that we have arrived at that particular level of accuracy. The required root is therefore $x = 2{\cdot}195823$ to 6 dp. Save the spreadsheet so that it can be used as a template for other such problems.

Now let us have another example.

Next frame

41

Example 2

The equation $x^3 + 2x^2 - 5x - 1 = 0$ is of the form $f(x) = 0$ where $f(1) < 0$ and $f(2) > 0$ so there is a solution to the equation between 1 and 2. We shall take this to be $x = 1{\cdot}5$. Use the Newton–Raphson method to find the root to six decimal places.

Use the previous spreadsheet as a template and make the following amendments

In cell B2 enter the number ....

42

$\boxed{1{\cdot}5}$

Because

That is the value of x_0 that is used to start the iteration

In cell C2 enter the formula ....

43

=B2^3 + 2*B2^2 – 5*B2 – 1

Because

That is the value of $f(x_0) = {x_0}^3 + 2{x_0}^2 - 5x_0 - 1$. Copy the contents of cell C2 into cells C3 to C5.

In cell D2 enter the formula ....

44

=3*B2^2 + 4*B2 – 5

Because

That is the value of $f'(x_0) = 3{x_0}^2 + 4x_0 - 5$. Copy the contents of cell D2 into cells D3 to D5.

In cell B2 the formula remains the same as ....

45

=B2 – C2/D2

The final display is then ....

46

n	x	$f(x)$	$f'(x)$
0	1·5	−0·625	7·75
1	1·580645	0·042798	8·817898
2	1·575792	0·000159	8·752524
3	1·575773	2·21E-09	8·75228

We cannot be sure that the value 1·575773 is accurate to the **sixth** decimal place so we must extend the table.

Highlight cells A5 to D5, click **Edit** on the Command bar and select **Copy** from the drop-down menu.

Place the cell highlight in cell A6, click **Edit** and then **Paste**.

The seventh row of the spreadsheet then fills to produce the display

n	x	$f(x)$	$f'(x)$
0	1·5	−0·625	7·75
1	1·580645	0·042798	8·817898
2	1·575792	0·000159	8·752524
3	1·575773	2·21E-09	8·75228
4	1·575773	−8·9E-16	8·75228

And the repetition of the x-value ensures that the solution $x = 1{\cdot}575773$ is indeed accurate to 6 dp.

Now do one completely on your own.

Next frame

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Example 3

The equation $2x^3 - 7x^2 - x + 12 = 0$ has a root near to $x = 1{\cdot}5$. Use the Newton–Raphson method to find the root to six decimal places.

The spreadsheet solution produces ....

48

$$\boxed{x = 1{\cdot}686141 \text{ to 6 dp}}$$

Because

Fill cells A2 to A6 with the numbers 0 to 4

In cell B2 enter the value for x_0, namely 1·5

In cell C2 enter the formula for $f(x_0)$, namely =2*B2^3 – 7*B2^2 – B2 + 12 and copy into cells C3 to C6

In cell D2 enter the formula for $f'(x_0)$, namely =6*B2^2 – 14*B2 – 1 and copy into cells D3 to D6

In cell B3 enter the formula for x_1, namely =B2 – C2/D2 and copy into cells B4 to B6.

The final display is ....

49

n	x	$f(x)$	$f'(x)$
0	1·5	1·5	−8·5
1	1·676471	0·073275	−7·60727
2	1·686103	0·000286	−7·54778
3	1·686141	4·46E-09	−7·54755
4	1·686141	0	−7·54755

As soon as the number in the second column is repeated then we know that we have arrived at that particular level of accuracy. The required root is therefore $x = 1{\cdot}686141$ to 6 dp.

First approximations

The whole process hinges on knowing a 'starter' value as first approximation. If we are not given a hint, this information can be found by either

(a) applying the remainder theorem if the function is a polynomial
(b) drawing a sketch graph of the function.

Example 4

Find the real root of the equation $x^3 + 5x^2 - 3x - 4 = 0$ correct to six significant figures.

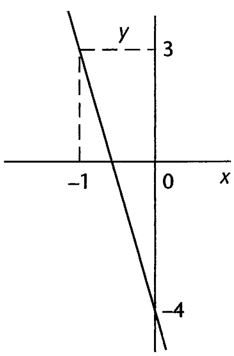
Application of the remainder theorem involves substituting $x = 0$, $x = \pm 1$, $x = \pm 2$, etc. until two adjacent values give a change in sign.

$$f(x) = x^3 + 5x^2 - 3x - 4$$

$$f(0) = -4; \quad f(1) = -1; \quad f(-1) = 3$$

The sign changes from $f(0)$ to $f(-1)$. There is thus a root between $x = 0$ and $x = -1$.

Therefore choose $x = -0{\cdot}5$ as the first approximation and then proceed as before.



Complete the table and obtain the root

$x = \ldots\ldots\ldots\ldots$

50

$$\boxed{x = -0{\cdot}675527}$$

The final spreadsheet display is

n	x	$f(x)$	$f'(x)$
0	−0·5	−1·375	−7·25
1	−0·689655	0·11907	−8·469679
2	−0·675597	0·000582	−8·386675
3	−0·675527	1·43E-08	−8·386262
4	−0·675527	0	−8·386262

51

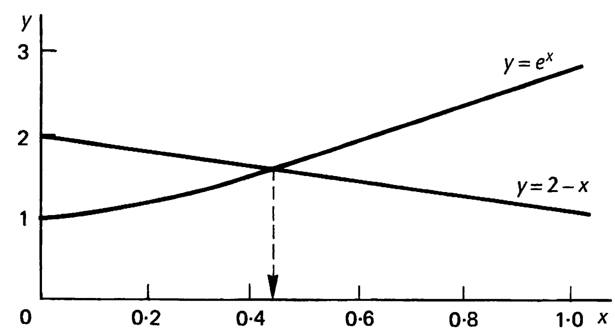
Example 5

Solve the equation $e^x + x - 2 = 0$ giving the root to 6 significant figures.

It is sometimes more convenient to obtain a first approximation to the required root from a sketch graph of the function, or by some other graphical means.

In this case, the equation can be rewritten as $e^x = 2 - x$ and we therefore sketch graphs of $y = e^x$ and $y = 2 - x$.

x	0·2	0·4	0·6	0·8	1
e^x	1·22	1·49	1·82	2·23	2·72
$2 - x$	1·8	1·6	1·4	1·2	1



It can be seen that the two curves cross over between $x = 0{\cdot}4$ and $x = 0{\cdot}6$.

Approximate root $x = 0{\cdot}4$

$$f(x) = e^x + x - 2 \qquad f'(x) = e^x + 1$$

$$x = \ldots\ldots\ldots\ldots$$

Finish it off

52

$$\boxed{x = 0{\cdot}442854}$$

The final spreadsheet display is

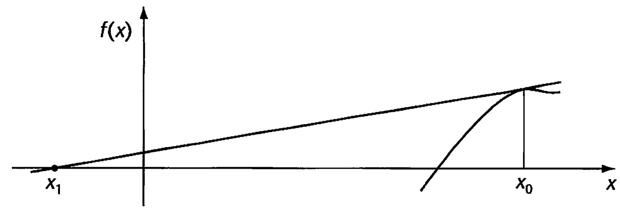
n	x	$f(x)$	$f'(x)$
0	0·4	−0·10818	2·491825
1	0·443412	0·001426	2·558014
2	0·442854	2·42E-07	2·557146
3	0·442854	7·11E-15	2·557146

Note: There are times when the normal application of the Newton–Raphson method fails to converge to the required root. This is particularly so when $f'(x_0)$ is very small, so before we leave this section let us consider this difficulty.

Modified Newton–Raphson method

53

If the slope of the curve at $x = x_0$ is small, the value of the second approximation $x = x_1$ may be further from the exact root at A than the first approximation.



If $x = x_0$ is an approximate solution of $f(x) = 0$ and $x = x_0 - h$ is the exact solution then $f(x_0 - h) = 0$. By Taylor's series

$$f(x_0 - h) = f(x_0) - hf'(x_0) + \frac{h^2}{2!}f''(x_0) - \ldots = 0$$

(a) If we assume that h is small enough to neglect terms of the order h^2 and higher then this equation can be written as

$f(x_0 - h) \approx f(x_0) - hf'(x_0)$, that is $f(x_0) - hf'(x_0) \approx 0$ and so

$h \approx \dfrac{f(x_0)}{f'(x_0)}$ giving $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$ as a better approximation to the solution of $f(x) = 0$.

This is, of course, the relationship we have been using and which may fail when $f'(x)$ is small.

Notice: h is positive unless the sign of $f(x_0)$ is the opposite of the sign of $f'(x_0)$.

(b) If we consider the first three terms then

$$f(x_0 - h) \approx f(x_0) - hf'(x_0) + \frac{h^2}{2!}f''(x_0) \approx 0, \text{ that is}$$

$$2f(x_0) - 2hf'(x_0) + h^2 f''(x_0) \approx 0$$

Since $f'(x_0)$ is small we shall assume that we can neglect it so

$$h = \pm\sqrt{\frac{-2f(x_0)}{f''(x_0)}}$$

That is $h = \sqrt{\dfrac{-2f(x_0)}{f''(x_0)}}$ unless the signs of $f(x_0)$ and $f'(x_0)$ are different when it is $h = -\sqrt{\dfrac{-2f(x_0)}{f''(x_0)}}$. We use this result only when $f'(x_0)$ is found to be very small. Having found x_1 from x_0 we then revert to the normal relationship $x_{n+1} = x_n - \dfrac{f(x_0)}{f'(x_0)}$ for subsequent iterations.

Note this

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Example 6

The equation $x^3 - 1{\cdot}3x^2 + 0{\cdot}4x - 0{\cdot}03 = 0$ is known to have a root near $x = 0{\cdot}7$. Determine the root to 6 significant figures.

We start off in the usual way.

$$f(x) = x^3 - 1{\cdot}3x^2 + 0{\cdot}4x - 0{\cdot}03$$
$$f'(x) = 3x^2 - 2{\cdot}6x + 0{\cdot}4$$

and complete the first line of the normal table.

n	x_n	$f(x_n)$	$f'(x_n)$	$h = \dfrac{f(x_n)}{f'(x_n)}$	$x_{n+1} = x_n - h$
0	0·7				

Complete just the first line of values.

55

We have

n	x_n	$f(x_n)$	$f'(x_n)$	$h = \dfrac{f(x_n)}{f'(x_n)}$	$x_{n+1} = x_n - h$
0	0·7	−0·044	0·05	−0·88	1·58

We notice at once that

(a) The value of x_1 is well away from the approximate value (0·7) of the root.

(b) The value of $f'(x_0)$ is small, i.e. 0·05.

To obtain x_1 we therefore make a fresh start, using the modified relationship $x_1 = \ldots\ldots\ldots\ldots$

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$$\boxed{x_1 = x_0 \pm \sqrt{\frac{-2f(x_0)}{f''(x_0)}}}$$

$$f(x) = x^3 - 1{\cdot}3x^2 + 0{\cdot}4x - 0{\cdot}03 = [(x - 1{\cdot}3)x + 0{\cdot}4]x - 0{\cdot}03$$
$$f'(x) = 3x^2 - 2{\cdot}6x + 0{\cdot}4 = (3x - 2{\cdot}6)x + 0{\cdot}4$$
$$f''(x) = 6x - 2{\cdot}6$$

n	x_0	$f(x_0)$	$f''(x_0)$	$h = \sqrt{\dfrac{-2f(x_0)}{f''(x_0)}}$	$x_1 = x_0 \pm h$
0	0·7	−0·044			

Complete the line

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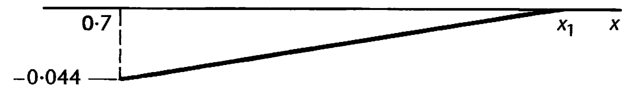
n	x_0	$f(x_0)$	$f''(x_0)$	$h=\sqrt{\dfrac{-2f(x_0)}{f''(x_0)}}$	$x_1=x_0\pm h$
0	0·7	−0·044	1·6	0·2345	0·9345

Note that in the expression $x_1 = x_0 \pm h$, we chose the positive sign since at $x_0 = 0{\cdot}7$, $f(x_0)$ is negative and the slope $f'(x_0)$ is positive.



Having established that $x_1 = 0{\cdot}9345$, we now revert to the usual $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ for the rest of the calculation. Complete the table therefore and obtain the required root.

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The final spreadsheet display is

n	x	$f(x)$	$f'(x)$	$f''(x)$
0	0·7	−0·044	0·05	1·6
1	0·934521	0·024625	0·590233	
2	0·892801	0·002544	0·469997	
3	0·887387	4·02E-05	0·45516	
4	0·887298	1·06E-08	0·454919	
5	0·887298	9·16E-16	0·454919	

Therefore to six decimal places the required root is $x = 0{\cdot}887298$.

Note that we only used the modified method to find x_1. After that the normal relationship is used.

And now ...

To date our task has been to find a value of x that satisfies an explicit equation $f(x) = 0$. This is quite general because *any* equation in x can be written in this form. For example, the equation

$$\sin x = x - e^{3x}$$

can always be written as

$$\sin x - x + e^{3x} = 0$$

and then approached by one of the methods that we have discussed so far.

What we want to do now is to work the other way – given a value of x, to find the corresponding value of $f(x)$. If $f(x)$ is given explicitly then this is no problem, it is just a matter of substituting the value of x in the formula and working it out. However, many times a function exists but it is not given explicitly, as in the case of a set of readings compiled as a result of an experiment or practical test. We shall consider this problem in the following frames.

Next frame

Interpolation

59 When a function is defined by a well-understood expression such as

$$f(x) = 4x^3 - 3x^2 + 7$$

or

$$f(x) = 5\sin(\exp[x])$$

the values of the dependent variable $f(x)$ corresponding to given values of the independent variable x can be found by direct substitution. Sometimes, however, a function is not defined in this way but by a collection of ordered pairs of numbers.

Example 1

A function can be defined by the following set of data:

x	$f(x)$
1	4
2	14
3	40
4	88
5	164
6	274

Intermediate values, for example, $x = 2{\cdot}5$, can be estimated by a process called **interpolation**.

The value of $f(2{\cdot}5)$ will clearly lie between 14 and 40, the function values for $x = 2$ and $x = 3$.

Purely as an estimate, $f(2{\cdot}5) = \ldots\ldots\ldots\ldots$

What do you suggest?

60 $\boxed{27}$

1 Linear interpolation

If you gave the result as 27, you no doubt agreed that $x = 2{\cdot}5$ is midway between $x = 2$ and $x = 3$, and that therefore $f(2{\cdot}5)$ would be midway between 14 and 40, i.e. 27. This is the simplest form of interpolation, but there is no evidence that there is a linear relationship between x and $f(x)$, and the result is therefore suspect.

Of course, we could have estimated the function value at $x = 2{\cdot}5$ by other means, such as

$\ldots\ldots\ldots\ldots$

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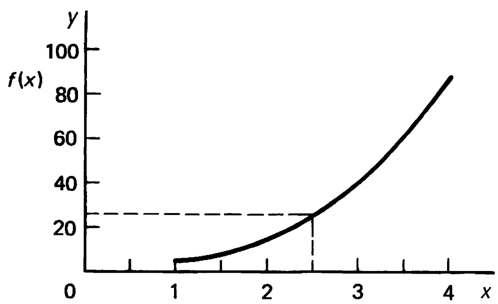
by drawing the graph of $f(x)$ against x

2 Graphical interpolation

We could, indeed, plot the graph of $f(x)$ against x and, from it, estimate the value of $f(x)$ at $x = 2{\cdot}5$.



This method is also approximate and time consuming.

$$f(2{\cdot}5) \approx 26$$

In what follows we shall look at interpolation using *finite differences*, which work well and quickly when the values of x are equally spaced. When the values of x are not equally spaced we need to resort to the more involved algebraic method called *Lagrangian interpolation* (which could also be used for equally spaced points).

Next frame

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3 Gregory–Newton interpolation formula using forward finite differences

x	$f(x)$
$\vdots$	$\vdots$
x_0	$f(x_0)$
x_1	$f(x_1)$
$\vdots$	$\vdots$

$$\Delta f_0 = f(x_1) - f(x_0)$$

We assume that $x_0, x_1, \ldots$ are distinct, equally spaced apart, and $x_0 < x_1 < \ldots$

For each pair of consecutive function values, $f(x_0)$ and $f(x_1)$, in the table, the *forward difference* Δf_0 is calculated by subtracting $f(x_0)$ from $f(x_1)$. This difference is written in a third column of the table, midway between the lines carrying $f(x_0)$ and $f(x_1)$.

x	$f(x)$	Δf
1	4	
		10
2	14	
		26
3	40	
$\vdots$	$\vdots$	

Complete the table for the data given in Frame 59 which then becomes

. . . .

63

x	$f(x)$	Δf
1	4	
		10
2	14	
		26
3	40	
		48
4	88	
		76
5	164	
		110
6	274	

We now form a fourth column, the forward differences of the values of Δf, denoted by $\Delta^2 f$, and again written midway between the lines of Δf. These are the second forward differences of $f(x)$.

So the table then becomes ....

64

x	$f(x)$	Δf	$\Delta^2 f$
1	4		
		10	
2	14		16
		26	
3	40		22
		48	
4	88		28
		76	
5	164		34
		110	
6	274		

A further column can now be added in like manner, giving the third differences, denoted by $\Delta^3 f$, so that we then have ....

65

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
1	4			
		10		
2	14		16	
		26		6
3	40		22	
		48		6
4	88		28	
		76		6
5	164		34	
		110		
6	274			

Notice that the table has now been completed, for the third differences are constant and all subsequent differences would be zero.

Now we shall see how to use the table. So move on

To find $f(2{\cdot}5)$ **66**

	x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
	1	4			
			10		
$x_0 \rightarrow$	2	14		16	
h x_p			26		6
$x_1 \rightarrow$	3	40		22	
			48		6
	4	88		28	
			76		6
	5	164		34	
			110		
	6	274			

We have to find $f(2{\cdot}5)$. Therefore denote $\left.\begin{array}{l} x = 2 \text{ as } x_0 \\ x = 3 \text{ as } x_1 \end{array}\right\} x = 2{\cdot}5 \text{ as } x_p$

Let h = the constant range between successive values of x,

i.e. $h = x_1 - x_0$

Express $(x_p - x_0)$ as a fraction of h, i.e. $p = \dfrac{x_p - x_0}{h}, \quad 0 < p < 1$

Therefore, in the case above, $h = 1$ and $p = \dfrac{2{\cdot}5 - 2{\cdot}0}{1} = 0{\cdot}5$.

All we now use from the table is the set of values underlined by the broken line drawn diagonally from $f(x_0)$.

So we have

$p = \ldots\ldots\ldots\ldots; \quad f_0 = \ldots\ldots\ldots\ldots; \quad \Delta f_0 = \ldots\ldots\ldots\ldots;$
$\Delta^2 f_0 = \ldots\ldots\ldots\ldots; \quad \Delta^3 f_0 = \ldots\ldots\ldots\ldots$

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$\boxed{p = 0{\cdot}5 \quad f_0 = 14; \quad \Delta f_0 = 26; \quad \Delta^2 f_0 = 22; \quad \Delta^3 f_0 = 6}$

Now we are ready to deal with the *Gregory–Newton forward difference interpolation formula*

$$f_p = f_0 + p\Delta f_0 + \frac{p(p-1)}{1 \times 2}\Delta^2 f_0 + \frac{p(p-1)(p-2)}{1 \times 2 \times 3}\Delta^3 f_0 + \ldots$$

This is sometimes written in operator form

$$f_p = \left\{1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \ldots\right\} f_0$$

which you no doubt recognise as the binomial expansion of

$$f_p = (1 + \Delta)^p \times f_0$$

Substituting the values in the above example gives

$$f(2{\cdot}5) = f_p = \ldots\ldots\ldots\ldots$$

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$$\boxed{24{\cdot}625}$$

Because

$$\begin{aligned} f_p &= 14 + 0{\cdot}5(26) + \frac{0{\cdot}5(-0{\cdot}5)}{1\times 2}(22) + \frac{0{\cdot}5(-0{\cdot}5)(-1{\cdot}5)}{1\times 2\times 3}(6) \\ &= 14 + 13 - 2{\cdot}75 + 0{\cdot}375 \\ &= 27{\cdot}375 - 2{\cdot}75 = 24{\cdot}625 \end{aligned}$$

Comparing the results of the three methods we have discussed

(a) Linear interpolation $f(2{\cdot}5) = 27$

(b) Graphical interpolation $f(2{\cdot}5) = 26$

(c) Gregory–Newton formula $f(2{\cdot}5) = 24{\cdot}625$ – the true value

Example 2

x	$f(x)$
2	14
4	88
6	274
8	620
10	1174

It is required to determine the value of $f(x)$ at $x = 5{\cdot}5$.

In this case

x_0 = ... x_1 = ...

h = ... p = ...

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$$\boxed{x_0 = 4; \quad x_1 = 6; \quad h = 2; \quad p = 0{\cdot}75}$$

Because

$$h = x_1 - x_0 = 6 - 4 = 2$$

$$p = \frac{x_p - x_0}{h} = \frac{5{\cdot}5 - 4}{2} = \frac{1{\cdot}5}{2} = 0{\cdot}75$$

First compile the table of forward differences ....

70

	x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
	2	14			
			74		
$x_0 \longrightarrow$	4	88		112	
			186		48
$x_1 \longrightarrow$	6	274		160	
			346		48
	8	620		208	
			554		
	10	1174			

The Gregory–Newton forward difference interpolation formula is

$$f_p = (1 + \Delta)^p \times f_0$$

i.e. $f_p =$ ....

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$$\boxed{\begin{aligned} f_p &= \left\{1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \dots\right\} f_0 \\ &= f_0 + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 f_0 + \dots \end{aligned}}$$

So, substituting the relevant values from the table, gives

$$f(5{\cdot}5) = f_p = \dots\dots\dots\dots$$

72

$$\boxed{214{\cdot}4}$$

Because

	x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
	2	14			
			74		
$x_0 \longrightarrow$	4	88		112	
$x_p \longrightarrow$			186		48
$x_1 \longrightarrow$	6	274		160	
			346		48
	8	620		208	
			554		
	10	1174			

$$\begin{aligned} f(5{\cdot}5) = f_p &= 88 + 0{\cdot}75(186) + \frac{0{\cdot}75(-0{\cdot}25)}{1 \times 2}(160) \\ &\quad + \frac{0{\cdot}75(-0{\cdot}25)(-1{\cdot}25)}{1 \times 2 \times 3}(48) \\ &= 88 + 139{\cdot}5 - 15 + 1{\cdot}875 = 214{\cdot}375 \end{aligned}$$

$$\therefore\ f(5{\cdot}5) = 214{\cdot}4$$

Finally, one more.

Example 3

Determine the value of $f(-1)$ from the set of function values.

x	-4	-2	0	2	4	6	8
$f(x)$	541	55	1	-53	-155	31	1225

Complete the working and then check with the next frame.

73

$$\boxed{f(-1) = 10}$$

Here is the working; method as before.

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
-4	541				
		-486			
$x_0 \to$ -2	55		432		
		-54		-432	
$x_1 \to$ 0	1		0		384
		-54		-48	
2	-53		-48		384
		-102		336	
4	-155		288		384
		186		720	
6	31		1008		
		1194			
8	1225				

$x_0 = -2$; $x_1 = 0$; $x_p = -1$; $\therefore\ h = 2$; $p = \frac{1}{2}$

$$f_p = f_0 + p\Delta f_0 + \frac{p(p-1)}{1 \times 2}\Delta^2 f_0 + \frac{p(p-1)(p-2)}{1 \times 2 \times 3}\Delta^3 f_0$$

$$+ \frac{p(p-1)(p-2)(p-3)}{1 \times 2 \times 3 \times 4}\Delta^4 f_0$$

$$= 55 + \frac{1}{2}(-54) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1 \times 2}(0) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1 \times 2 \times 3}(-48)$$

$$+ \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{1 \times 2 \times 3 \times 4}(384)$$

$$= 55 - 27 + 0 - 3 - 15 = 10$$

$$\therefore\ f_p = f(-1) = 10$$

This table of data does have its restrictions. For example, if we had wanted to find $f(2{\cdot}5)$ from the table we would have run out of data because there is no $\Delta^4 f$ entry available. In such a case we can resort to a zig-zag path through the table using **central differences**.

Next frame

Central differences

74

The central difference operator δ is defined by its action on the expression $f(x)$ as

$$\delta f(x) = f(x + h/2) - f(x - h/2)$$

and using this operator the interpolated value of $f(x)$ near to the given value of f_0 is defined by the **Gauss forward** formula as

$$f_p = f_0 + p\delta f_{0+\frac{1}{2}} + \frac{p(p-1)}{2!}\delta^2 f_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 f_{0+\frac{1}{2}} + \frac{(p+1)p(p-1)(p-2)}{4!}\delta^4 f_0 + \dots$$

or by the **Gauss backward** formula as

$$f_p = f_0 + p\delta f_{0-\frac{1}{2}} + \frac{(p+1)p}{2!}\delta^2 f_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 f_{0-\frac{1}{2}} + \frac{(p+2)(p+1)p(p-1)}{4!}\delta^4 f_0 + \dots$$

There are no tabulated values at the half-interval values $x_0 + h/2$ and $x_0 - h/2$ and so these are taken to be the differences evaluated at mid-interval as given in the forward difference table. This means that the tables for the Gregory–Newton forward differences and the central differences are identical (apart, that is, from the column headings); the method of tracing through the table, however, is different. For example, to find $f(2{\cdot}5)$ for the example given in Frame 59

x	$f(x)$	$\delta f(x)$	$\delta^2 f(x)$	$\delta^3 f(x)$
1	4			
		10		
2	14		16	
		26		6
3	40		22	
		48		6
4	88		28	
		76		6
5	164		34	
		110		
6	274			

Here $x_0 = 2$, $f_0 = 14$, $\delta f_{0+\frac{1}{2}} = 26$, $\delta^2 f_0 = 16$, $\delta^3 f_{0+\frac{1}{2}} = 6$, $\delta^4 f_0 = 0$ and $p = 0{\cdot}5$. Thus

$$\begin{aligned} f_p &= 14 + (0{\cdot}5)26 + \frac{(0{\cdot}5)(-0{\cdot}5)}{2}16 + \frac{(0{\cdot}5)(-0{\cdot}5)(-1{\cdot}5)}{6}6 \\ &= 14 + 13 - 2 - 0{\cdot}375 = 24{\cdot}625 \end{aligned}$$

which agrees with the value found using the Gregory–Newton forward difference formula.

▶

Try one for yourself. The given tabulated values are

x	$f(x)$	$\delta f(x)$	$\delta^2 f(x)$	$\delta^3 f(x)$
0	-5			
		3		
1	-2		6	
		9		12
2	7		18	
		27		12
3	34		30	
		57		
4	91			

Using the Gauss forward difference formula, the interpolated value of

$f(2{\cdot}2) = \ldots\ldots\ldots\ldots$

Next frame

75

$\boxed{10{\cdot}576}$

Because

Using $f_p = f_0 + p\delta f_{0+\frac{1}{2}} + \dfrac{p(p-1)}{2!}\delta^2 f_0 + \dfrac{p(p-1)(p+1)}{3!}\delta^3 f_{0+\frac{1}{2}} + \ldots$ and following the solid line through the table where

$x_0 = 2,\quad f_0 = 7,\quad \delta f_{0+\frac{1}{2}} = 27,\quad \delta^2 f_0 = 18,\quad \delta^3 f_{0+\frac{1}{2}} = 12$ and $p = 0{\cdot}2$,

$$\text{then } f_p = 7 + (0{\cdot}2)27 + \frac{(0{\cdot}2)(-0{\cdot}8)}{2}18 + \frac{(0{\cdot}2)(-0{\cdot}8)(1{\cdot}2)}{6}12$$

$$= 7 + 5{\cdot}4 - 1{\cdot}44 - 0{\cdot}384$$

$$= 10{\cdot}576$$

Using the Gauss backward difference formula (following the broken line)

$$f_p = f_0 + p\delta f_{0-\frac{1}{2}} + \frac{p(p+1)}{2!}\delta^2 f_0 + \frac{p(p-1)(p+1)}{3!}\delta^3 f_{0-\frac{1}{2}} + \ldots$$

where here $\delta f_{0-\frac{1}{2}} = 9$ and $\delta^3 f_{0-\frac{1}{2}} = 12$ and so

$$f_p = 7 + (0{\cdot}2)9 + \frac{(0{\cdot}2)(1{\cdot}2)}{2}18 + \frac{(0{\cdot}2)(1{\cdot}2)(-0{\cdot}8)}{6}12$$
$$= 7 + 1{\cdot}8 + 2{\cdot}16 - 0{\cdot}384 = 10{\cdot}576$$

as found with the Gauss forward difference formula.

Next frame

Gregory–Newton backward differences

76

We have seen that the Gregory–Newton forward difference procedure loses terms if the interpolation is for points sufficiently forward in the table. We have also seen how this difficulty can be avoided by using central differences. However, even with central differences we can run out of data before completing a full traverse of the table. In such a situation we resort to the Gregory–Newton backward difference formula

$$f_p = f_0 + p\Delta f_{-1} + \frac{p(p+1)}{2!}\Delta^2 f_{-2} + \frac{p(p+1)(p+2)}{3!}\Delta^3 f_{-3} + \ldots$$

As an example, consider the table of Frame 74.

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
1	4			
		10		
2	14		16	
		26		6
3	40		22	
		48		6
4	88		28	
		76		6
5	164		34	
		110		
6	274			

Using this table we can calculate $f(5{\cdot}5)$ by tracing back through the table (see broken line) as

$$\begin{aligned} f(5{\cdot}5) &= f_0 + (0{\cdot}5)\Delta f_{-1} + \frac{(0{\cdot}5)(1{\cdot}5)}{2}\Delta^2 f_{-2} + \frac{(0{\cdot}5)(1{\cdot}5)(2{\cdot}5)}{6}\Delta^3 f_{-3} \\ &= 164 + (0{\cdot}5)76 + \frac{(0{\cdot}5)(1{\cdot}5)28}{2} + \frac{(0{\cdot}5)(1{\cdot}5)(2{\cdot}5)6}{6} \\ &= 214{\cdot}375 \end{aligned}$$

77

In each of the examples that we have looked at so far the tabular display of differences eventually results in a column of zeros and this determines the number of terms in an interpolation calculation. The zeros have arisen because all the examples have been derived from polynomials. The following example deals with a tabular display of differences which does not result in a column of zeros. In this case the number of terms used in the interpolation calculation determines confidence in the accuracy of the result.

Example

Use the Gregory–Newton forward difference method to find $f(0{\cdot}15)$ to 4 decimal places from the following finite difference table

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
0	0·000000			
		0·099833		
0·1	0·099833		−0·000998	
		0·098836		−0·000988
0·2	0·198669		−0·001985	
		0·096851		−0·000968
0·3	0·295520		−0·002953	
		0·093898		−0·000938
0·4	0·389418		−0·003891	
		0·090007		
0·5	0·479426			

Here $x_0 = 0{\cdot}1$, $x_1 = 0{\cdot}2$, $x_p = 0{\cdot}15$ and therefore $p = 0{\cdot}5$, and

$$
\begin{aligned}
f_p &= f_0 + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 f_0 + \dots \\
&= 0{\cdot}099833 + \frac{1}{2}(0{\cdot}098836) + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(-0{\cdot}001985)/2 \\
&\quad + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-0{\cdot}000969)/6 + \dots \\
&= 0{\cdot}099833 + 0{\cdot}049418 + 0{\cdot}000248 - 0{\cdot}000061 + \dots \\
&= 0{\cdot}1494 \text{ to 4 dp}
\end{aligned}
$$

As you can see, the calculation can continue indefinitely and termination is dictated by the number of decimal places required in the final answer.

Lagrange interpolation

78

If the straight line $p(x) = a_0 + a_1 x$ passes through the two points $(x_0, f(x_0))$ and $(x_1, f(x_1))$, where a_0 and a_1 are constants, then the equation for this line can also be written as

$$p(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

For example, the straight line $p(x) = 3 + 2x$ passes through the two points (1, 5) and (2, 7). Substituting the values for the variables in the above equation demonstrates this alternative form for the equation

$$p(x) = \frac{x - 2}{1 - 2} 5 + \frac{x - 1}{2 - 1} 7 = 10 - 5x + 7x - 7 = 3 + 2x$$

So, given the two data points from Frame 59, (2, 14) and (3, 40), using linear interpolation

$$f(2{\cdot}5) \approx p(2{\cdot}5) = \dots\dots\dots\dots$$

79

$$\boxed{27}$$

Because

$$\begin{aligned} p(x) &= \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1) \\ &= \frac{x - 3}{2 - 3} 14 + \frac{x - 2}{3 - 2} 40 = 26x - 38 \end{aligned}$$

and so

$$f(2{\cdot}5) \approx p(x) = 26(2{\cdot}5) - 38 = 27$$

80

The principle of Lagrange interpolation is that a function $f(x)$ whose values are given at a collection of points is assumed to be approximately represented by a polynomial $p(x)$ that passes through each and every point. The polynomial is called the **interpolation polynomial** and it is of degree one less than the number of points given. For two data points the interpolating polynomial is taken to be a linear polynomial, as you have just seen in the last example. For three data points the interpolating polynomial is taken to be a quadratic, for four data points the interpolation polynomial is taken to be a cubic, and so on.

▶

In the same manner as before it can be shown that the quadratic

$$p(x) = a_0 + a_1 x + a_2 x^2$$

that passes through the three points $(x_0, f(x_0))$, $(x_1, f(x_1))$ and $(x_2, f(x_2))$ can be written as

$$p(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

So let's try one. Given the collection of values

x	$f(x)$
1·5	0·405
2·1	0·742
3	1·099

by Lagrangian interpolation, $f(1{\cdot}8) \approx$ .... to 2 decimal places

81

$$\boxed{0{\cdot}58}$$

Because

$$\begin{aligned}
p(x) &= \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) \\
&\quad + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2) \\
&= \frac{(x - 2{\cdot}1)(x - 3)}{(1{\cdot}5 - 2{\cdot}1)(1{\cdot}5 - 3)} 0{\cdot}405 + \frac{(x - 1{\cdot}5)(x - 3)}{(2{\cdot}1 - 1{\cdot}5)(2{\cdot}1 - 3)} 0{\cdot}742 \\
&\quad + \frac{(x - 1{\cdot}5)(x - 2{\cdot}1)}{(3 - 1{\cdot}5)(3 - 2{\cdot}1)} 1{\cdot}099 \\
&= \frac{(x^2 - 5{\cdot}1x + 6{\cdot}3)}{0{\cdot}9} 0{\cdot}405 + \frac{(x^2 - 4{\cdot}5x + 4{\cdot}5)}{(-0{\cdot}54)} 0{\cdot}742 \\
&\quad + \frac{(x^2 - 3{\cdot}6x + 3{\cdot}15)}{1{\cdot}35} 1{\cdot}099 \\
&= -0{\cdot}11x^2 + 0{\cdot}958x - 0{\cdot}784
\end{aligned}$$

So that

$$f(1{\cdot}8) \approx p(1{\cdot}8) = 0{\cdot}58 \text{ to 2 decimal places.}$$

By carefully considering the interpolating polynomials for two and three data points you should be able to see a pattern. Write down what you think the interpolating polynomial should be for four data points:

....

82

$$\boxed{\begin{aligned}p(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)\\&+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)\end{aligned}}$$

Use this interpolating polynomial for the data points

x	$f(x)$
1	0·368
1·2	0·301
1·3	0·273
1·5	0·223

To 2 decimal places, $f(1{\cdot}4) \approx$ ....

83

$\boxed{0{\cdot}25}$

Because $p(x)$

$$\begin{aligned}&= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)\\&\quad+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)\\&= \frac{(x-1{\cdot}2)(x-1{\cdot}3)(x-1{\cdot}5)}{(1-1{\cdot}2)(1-1{\cdot}3)(1-1{\cdot}5)}0{\cdot}368+\frac{(x-1)(x-1{\cdot}3)(x-1{\cdot}5)}{(1{\cdot}2-1)(1{\cdot}2-1{\cdot}3)(1{\cdot}2-1{\cdot}5)}0{\cdot}301\\&\quad+\frac{(x-1)(x-1{\cdot}2)(x-1{\cdot}5)}{(1{\cdot}3-1)(1{\cdot}3-1{\cdot}2)(1{\cdot}3-1{\cdot}5)}0{\cdot}273+\frac{(x-1)(x-1{\cdot}2)(x-1{\cdot}3)}{(1{\cdot}5-1)(1{\cdot}5-1{\cdot}2)(1{\cdot}5-1{\cdot}3)}0{\cdot}223\\&= \frac{(x^3-4x^2+5{\cdot}31x-2{\cdot}34)}{(-0{\cdot}03)}0{\cdot}368+\frac{(x^3-3{\cdot}8x^2+4{\cdot}75x-1{\cdot}95)}{0{\cdot}006}0{\cdot}301\\&\quad+\frac{(x^3-3{\cdot}7x^2+4{\cdot}5x-1{\cdot}8)}{(-0{\cdot}006)}0{\cdot}273+\frac{(x^3-3{\cdot}5x^2+4{\cdot}06x-1{\cdot}56)}{0{\cdot}03}0{\cdot}223\\&= -0{\cdot}167x^3+0{\cdot}767x^2-1{\cdot}415x+1{\cdot}183\end{aligned}$$

So that

$f(1{\cdot}4) \approx p(1{\cdot}4) = 0{\cdot}25$ to 2 decimal places

The general Lagrange interpolation polynomial for $n+1$ data points at $x_0, x_1, \ldots, x_n$ is

$$\begin{aligned}p(x) &= \frac{(x-x_1)(x-x_2)(\cdots)(x-x_n)}{(x_0-x_1)(x_0-x_2)(\cdots)(x_0-x_n)}f(x_0)\\&\quad+\frac{(x-x_0)(x-x_2)(\cdots)(x-x_n)}{(x_1-x_0)(x_1-x_2)(\cdots)(x_1-x_n)}f(x_1)+\cdots\\&\quad\cdots+\frac{(x-x_0)(x-x_1)(\cdots)(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(\cdots)(x_n-x_{n-1})}f(x_n)\end{aligned}$$

▶

This now completes the work of this Programme. What follows is a **Revision summary** and a **Can You?** checklist. Read the summary carefully and respond to the questions in the checklist. When you feel sure that you are happy with the content of this Programme, try the **Test exercise**. Take your time, there is no need to hurry. Finally, a collection of **Further problems** provides valuable additional practice.

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Revision summary 1

1 The *Fundamental Theorem of Algebra* can be stated as follows:
Every polynomial expression $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ can be written as a product of n linear factors in the form

$$f(x) = a_n(x - r_1)(x - r_2)(\cdots)(x - r_n)$$

2 *Relations between the coefficients and the roots of a polynomial equation*
Whenever a polynomial with *real coefficients* a_i has a complex root it also has the complex conjugate as another root.
If α, β, $\gamma, \ldots$ are the roots of the equation

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0$$

then, provided $p_0 \neq 0$

sum of roots $= -\dfrac{p_1}{p_0}$

sum of the product of the roots, taken two at a time $= \dfrac{p_2}{p_0}$

sum of the product of the roots, taken three at a time $= -\dfrac{p_3}{p_0}$

sum of the product of the roots, taken n at a time $= (-1)^n \dfrac{p_n}{p_0}$.

3 *Cubic equations*
Every cubic equation with real coefficients has at least one real root that can be found by **Tartaglia's solution**. The real root of $x^3 + ax + b = 0$, $a > 0$ is

$$x = \left\{-\frac{b}{2} + \sqrt{\frac{a^3}{27} + \frac{b^2}{4}}\right\}^{1/3} + \left\{-\frac{b}{2} - \sqrt{\frac{a^3}{27} + \frac{b^2}{4}}\right\}^{1/3}$$

Reduced form
Every cubic equation of the form $x^3 + ax^2 + bx + c = 0$ can be written in reduced form $y^3 + py + q = 0$ by using the transformation $x = y - \dfrac{a}{3}$.

4 *Numerical methods*
Bisection
The bisection method of finding a solution to the equation $f(x) = 0$ consists of

Finding a value of x such that $f(x) < 0$, say $x = a$
Finding a value of x such that $f(x) > 0$, say $x = b$.

The solution to the equation $f(x) = 0$ must then lie between a and b. Furthermore, it must lie either in the first half of the interval between a and b or in the second half.

▶

5 *Numerical solution of equations by iteration*
The process of finding the numerical solution to the equation

$$f(x) = 0$$

by iteration is performed by first finding an approximate solution and then using this approximate solution to find a more accurate solution. This process is repeated until a solution is found to the required level of accuracy.

6 *Using a spreadsheet*
Iteration procedures are more efficiently performed using a spreadsheet.

7 *Newton–Raphson iteration method*
If $x = x_0$ is an approximate solution to the equation $f(x) = 0$, a better approximation $x = x_1$ is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, \text{ and in general } x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

8 *Modified Newton–Raphson iteration method*
If, in the Newton–Raphson procedure $f'(x_0)$ is sufficiently small enough to cause the value of x_1 to be a worse approximation to the solution than x_0, then x_1 is obtained from the relationship

$$x_1 = x_0 \pm \sqrt{\frac{-2f(x_0)}{f''(x_0)}}$$

Subsequent iterations then use $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

9 *Interpolation*
Linear
Graphical

10 *Gregory–Newton interpolation formulas using central finite differences*

$$f_p = f_0 + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 f_o + \cdots$$

11 *Gauss interpolation formulas using central finite differences*
Gauss forward formula

$$f_p = f_0 + p\delta f_{0+\frac{1}{2}} + \frac{p(p-1)}{2!}\delta^2 f_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 f_{0+\frac{1}{2}} + \frac{(p+1)p(p-1)(p-2)}{4!}\delta^4 f_0 + \cdots$$

Gauss backward formula

$$f_p = f_0 + p\delta f_{0-\frac{1}{2}} + \frac{(p+1)p}{2!}\delta^2 f_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 f_{0-\frac{1}{2}} + \frac{(p+2)(p+1)p(p-1)}{4!}\delta^4 f_0 + \cdots$$

▶

12 *Gregory–Newton interpolation formula using backward finite differences*

$$f_p = f_0 + p\Delta f_{-1} + \frac{p(p+1)}{2!}\Delta^2 f_{-2} + \frac{p(p+1)(p+2)}{3!}\Delta^3 f_{-3} + \cdots$$

13 *Lagrange interpolation*

If the straight line $p(x) = a_0 + a_1 x$ passes through the two points $(x_0, f(x_0))$ and $(x_1, f(x_1))$, where a_0 and a_1 are constants, then the interpolation polynomial (straight line) for this line can be written as

$$p(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

The quadratic interpolating polynomial that passes through the three points $(x_0, f(x_0))$, $(x_1, f(x_1))$ and $(x_2, f(x_2))$ can be written as

$$\begin{aligned} p(x) = {} & \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) \\ & + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2) \end{aligned}$$

The cubic interpolating polynomial that passes through the four data points $(x_0, f(x_0))$, $(x_1, f(x_1))$, $(x_2, f(x_2))$ and $(x_3, f(x_3))$ can be written as

$$\begin{aligned} p(x) = {} & \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} f(x_0) \\ & + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} f(x_1) \\ & + \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} f(x_2) \\ & + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} f(x_3) \end{aligned}$$

The interpolating polynomial that passes through $n + 1$ data points is

$$\begin{aligned} p(x) = {} & \frac{(x - x_1)(x - x_2)(\cdots)(x - x_n)}{(x_0 - x_1)(x_0 - x_2)(\cdots)(x_0 - x_n)} f(x_0) \\ & + \frac{(x - x_0)(x - x_2)(\cdots)(x - x_n)}{(x_1 - x_0)(x_1 - x_2)(\cdots)(x_1 - x_n)} f(x_1) + \cdots \\ & \cdots + \frac{(x - x_0)(x - x_1)(\cdots)(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)(\cdots)(x_n - x_{n-1})} f(x_n) \end{aligned}$$

Can You?

Checklist 1

85

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Appreciate the Fundamental Theorem of Algebra? 1 to 3

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Find the two roots of a quadratic equation and recognise that for polynomial equations with real coefficients complex roots exist in complex conjugate pairs? 4 to 6

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Use the relationships between the coefficients and the roots of a polynomial equation to find the roots of the polynomial? 7 to 17

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Transform a cubic equation to reduced form? 18 to 20

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Use Tartaglia's solution to find the real root of a cubic equation? 21 and 22

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Find the solution of the equation $f(x) = 0$ by the method of bisection? 23 to 26

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Solve equations involving a single real variable by iteration and use a spreadsheet for efficiency? 27 to 33

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Solve equations using the Newton–Raphson iterative method? 34 to 52

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Use the modified Newton–Raphson method to find the first approximation when the derivative is small? 53 to 58

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Understand the meaning of interpolation and use simple linear and graphical interpolation? 59 to 61

  Yes ☐ ☐ ☐ ☐ ☐ *No*

▶

- Use the Gregory–Newton interpolation formula using forward and backward differences for equally spaced domain points? 62 to 73

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Use the Gauss interpolation formulas using central differences for equally spaced domain points? 74 to 77

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Use Lagrange interpolation when the domain points are not equally spaced? 78 to 83

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 1

86

1 Given that $x = -1 + j\sqrt{3}$ is one root of a quadratic equation with real coefficients, find the other root and hence the quadratic equation.

2 Solve the cubic equation $2x^3 - 7x^2 - 42x + 72 = 0$.

3 Write the cubic $3x^3 + 5x^2 + 3x + 5$ in reduced form and use Tartaglia's method to find the real root.

4 Use the method of bisection to find a solution to $x^3 - 5 = 0$ correct to 4 significant figures.

5 Use the Newton–Raphson method to find a positive solution of the following equation, correct to 6 decimal places:

$$\cos 3x = x^2$$

6 Use the modified Newton–Raphson method to find the solution correct to 6 decimal places near to $x = 2$ of the equation

$$x^3 - 6x^2 + 13x - 9 = 0$$

7 Given the table of values

x	$f(x)$
1	0
2	19
3	70
4	171
5	340
6	595

estimate

(a) $f(2{\cdot}5)$ using the Gregory–Newton forward difference formula
(b) $f(3{\cdot}4)$ using the Gauss central difference formula
(c) $f(5{\cdot}6)$ using the Gregory–Newton backward difference formula. ▶

8 Given the table of values

x	$f(x)$
1	4
2	−9
5	−108

use Lagrangian interpolation to estimate the value of $f(2{\cdot}2)$.

Further problems 1

1 Given that $x = \dfrac{-1 - j\sqrt{3}}{2}$ and $x = \dfrac{-1 + j}{\sqrt{2}}$ are two roots of a quartic equation with real coefficients, find the other two roots and hence the quartic equation. 87

2 Solve the equation $x^3 - 5x^2 - 8x + 12 = 0$, given that the sum of two of the roots is 7.

3 Find the values of the constants p and q such that the function $f(x) = 2x^3 + px^2 + qx + 6$ may be exactly divisible by $(x - 2)(x + 1)$.

4 If $f(x) = 4x^4 + px^3 - 23x^2 + qx + 11$ and when $f(x)$ is divided by $2x^2 + 7x + 3$ the remainder is $3x + 2$, determine the values of p and q.

5 If one root of the equation $x^3 - 2x^2 - 9x + 18 = 0$ is the negative of another, determine the three roots.

6 Solve the equation $x^3 - 7x^2 - 21x + 27 = 0$, given that the roots form a geometric sequence.

7 Form the equation whose roots are those of the equation $x^3 + x^2 + 9x + 9 = 0$ each increased by 2.

8 Form the equation whose roots exceed by 3 the roots of the equation $x^3 - 4x^2 + x + 6 = 0$.

9 If the equation $4x^3 - 4x^2 - 5x + 3 = 0$ is known to have two roots whose sum is 2, solve the equation.

10 Solve the equation $x^3 - 10x^2 + 8x + 64 = 0$, given that the product of two of the roots is the negative of the third.

11 Form the equation whose roots exceed those of the equation $2x^3 - 3x^2 - 11x + 6 = 0$ by 2.

12 If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, prove that $\alpha^2 + \beta^2 + \gamma^2 = p^2 - 2q$.

13 Using Tartaglia's solution, find the real root of the equation $2x^3 + 4x - 5 = 0$ giving the result to 4 significant figures.

14 Solve the equation $x^3 - 6x - 4 = 0$.

15 Rewrite the equation $x^3 + 6x^2 + 9x + 4 = 0$ in reduced form and hence determine the three roots.

16 Show that the equation $x^3 + 3x^2 - 4x - 6 = 0$ has a root between $x = 1$ and $x = 2$, and use the Newton–Raphson iterative method to evaluate this root to 4 significant figures.

17 Find the real root of the equations:

(a) $x^3 + 4x + 3 = 0$ (b) $5x^3 + 2x - 1 = 0$.

18 Solve the following equations:

(a) $x^3 - 5x + 1 = 0$ (b) $x^3 + 2x - 3 = 0$
(c) $x^3 - 4x + 1 = 0$.

19 Express the following in reduced form and determine the roots:

(a) $x^3 + 6x^2 + 9x + 5 = 0$
(b) $8x^3 + 20x^2 + 6x - 9 = 0$
(c) $4x^3 - 9x^2 + 42x - 10 = 0$.

20 Use the Newton–Raphson iterative method to solve the following.

(a) Show that a root of the equation $x^3 + 3x^2 + 5x + 9 = 0$ occurs between $x = -2$ and $x = -3$. Evaluate the root to four significant figures.

(b) Show graphically that the equation $e^{2x} = 25x - 10$ has two real roots and find the larger root correct to four significant figures.

(c) Verify that the equation $x - \cos x = 0$ has a root near to $x = 0{\cdot}8$ and determine the root correct to three significant figures.

(d) Obtain graphically an approximate root of the equation $2\ln x = 3 - x$. Evaluate the root correct to four significant figures.

(e) Verify that the equation $x^4 + 5x - 20 = 0$ has a root at approximately $x = 1{\cdot}8$. Determine the root correct to five significant figures.

(f) Show that the equation $x + 3\sin x = 2$ has a root between $x = 0{\cdot}4$ and $x = 0{\cdot}6$. Evaluate the root correct to five significant figures.

(g) The equation $2\cos x = e^x - 1$ has a real root between $x = 0{\cdot}8$ and $x = 0{\cdot}9$. Evaluate the root correct to four significant figures.

(h) The equation $20x^3 - 22x^2 + 5x - 1 = 0$ has a root at approximately $x = 0{\cdot}6$. Determine the value of the root correct to four significant figures.

▶

21 A polynomial function is defined by the following set of function values

x	2	4	6	8	10
$y = f(x)$	$-7{\cdot}00$	$9{\cdot}00$	$97{\cdot}0$	305	681

Find

(a) $f(4{\cdot}8)$ using the Gregory–Newton forward difference formula

(b) $f(7{\cdot}2)$ using the Gauss central difference formula

(c) $f(8{\cdot}5)$ using the Gregory–Newton backward difference formula.

22 For the function $f(x)$

x	4	5	6	7	8	9	10
$f(x)$	-10	12	56	128	234	380	572

Find

(a) $f(4{\cdot}5)$ and $f(6{\cdot}4)$ using the Gregory–Newton forward difference formula

(b) $f(7{\cdot}1)$ and $f(8{\cdot}9)$ using the Gregory–Newton backward difference formula.

23

x	2	4	6	8	10	12
$f(x)$	-9	35	231	675	1463	2691

For the function defined in the table above, evaluate (a) $f(2{\cdot}6)$ and (b) $f(7{\cdot}2)$.

24 A function $f(x)$ is defined by the following table

x	-4	-2	0	2	4	6	8
$f(x)$	277	51	1	-17	-147	-533	-1319

Find

(a) $f(-3)$ and $f(1{\cdot}6)$ using the Gregory–Newton forward difference formula

(b) $f(0{\cdot}2)$ and $f(3{\cdot}1)$ using the Gauss central difference formula

(c) $f(4{\cdot}4)$ and $f(7)$ using the Gregory–Newton backward difference formula.

▶

25 Given the table of values

x	$f(x)$
-1	$-2{\cdot}71828$
3	$-0{\cdot}04979$
5	$-0{\cdot}00674$

use Lagrangian interpolation to find the value of $f(3{\cdot}4)$.

26 Given the table of values

x	$f(x)$
6	$0{\cdot}801153$
$7{\cdot}2$	$-0{\cdot}82236$
9	$-0{\cdot}73922$
13	$0{\cdot}994808$

use Lagrangian interpolation to find the value of $f(8)$.

27 Given the table of values

x	$f(x)$
-2	$-2{\cdot}63906$
0	$-2{\cdot}48491$
5	$-1{\cdot}94591$
6	$-1{\cdot}79176$

use Lagrangian interpolation to find the values of

(a) $f(-0{\cdot}8)$

(b) $f(0{\cdot}8)$

(c) $f(5{\cdot}5)$.